Instanton representation of Plebanski gravity III: Classical constraints algebra

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Abstract

We compute the classical algebra of constraints for the instanton representation of Plebanski gravity. The constraints are first class in this representation, with a slightly different structure than in the Ashtekar variables and the metric representation. One main result is that the Hamiltonian constraint forms its own subalgebra in the instanton representation, which has implications for its implementation at the quantum level of the theory.

1 Introduction: Constraints algebra of GR

Denote by $\boldsymbol{A}_{Diff}$, the algebra of constraints corresponding to diffeomorphism invariance of general relativity. In the metric description of gravity $\boldsymbol{A}_{Diff}$ is given by

$$\begin{aligned}\{\vec{H}[\vec{N}],\vec{H}[\vec{M}]\} &= H_k\big[N^i\partial^k M_i - M^i\partial^k N_i\big];\\ \{H(\underline{N}),\vec{H}[\vec{N}]\} &= H[N^i\partial_i\underline{N}];\\ \big[H(\underline{N}),H(\underline{M})\big] &= H_i[\big(\underline{N}\partial_j\underline{M} - \underline{M}\partial_j\underline{N}\big)H^{ij}].\end{aligned} \tag{1}$$

$H[N]$ and $H_i[N^i]$ are the Hamiltonian and diffeomorphism constraints, smeared by the lapse function and the shift vector $N^\mu = (N, N^i)$ which are auxiliary fields. Seen as an algebra of spacetime diffeomorphisms $H_\mu = (H, H_i)$, equation (1) is first class in the Dirac sense [1] due to closure of the algebra. Within (1) one sees that the spatial diffeomorphisms H_i form a first class subalgebra of their own $\boldsymbol{A}_{diff} \subset \boldsymbol{A}_{Diff}$, but do not form an ideal within $\boldsymbol{A}_{Diff}$. If one started with a theory purely of spatial diffeomorphisms, then the algebra $\boldsymbol{A}_{diff}$ would be first class. However, a theory based entirely on the Hamiltonian constraint would be second class, since the Poisson bracket of two Hamiltonian constraints does not close into a Hamiltonian constraint.[1]

In the Ashtekar complex formalism of general relativity [2], [3], gravity is treated as a $SU(2)_-$ gauge theory with additional constraints H_μ. The initial value constraints are given by (H, H_i, G_a) which appends a triple of additional constraints to (1), the Gauss' law constraint G_a, due to $SU(2)_-$ gauge invariance.[2] Independently of general relativity, G_a by itself forms a first class Lie algebra

$$\{G_a[\theta^a], G_b[\lambda^b]\} = G_a\big[f^a_{bc}\theta^b\lambda^c\big], \tag{2}$$

where f_{abc} are the $SU(2)_-$ structure constants in the adjoint representation. The effect of the Ashtekar formalism is to embed the phase space of general relativity into this $SU(2)_-$ gauge theory, thus enlarging its algebraic structure. For consistency, one must verify closure the new system by computing the Poisson brackets of G_a with H_i and H. This yields as

[1]This is a new interpretation, the rationale for which will become clear when we compare $\boldsymbol{A}_{Diff}$ with the algebra of constraints in the instanton representation of GR.

[2]In this paper, symbols from the beginning of the Latin alphabet $a, b, c, \dots$ are used to denote internal $SU(2)_-$ indices and those from the middle $1, j, k, \dots$ are used for spatial indices.

an enlarged algebra the semidirect product of gauge transformations and spacetime diffeomorphisms $\boldsymbol{A}_{gauge} \times \boldsymbol{A}_{Diff}$, given by [2],[3]

$$\begin{aligned}
\{\vec{H}[\vec{N}], \vec{H}[\vec{M}]\} &= H_k\big[N^i \partial^k M_i - M^i \partial^k N_i\big]; \\
\{\vec{H}[N], G_a[\theta^a]\} &= G_a[N^i \partial_i \theta^a]; \\
\{G_a[\theta^a], G_b[\lambda^b]\} &= G_a\big[f^a_{bc} \theta^b \lambda^c\big]; \\
\{H(\underline{N}), \vec{H}[\vec{N}]\} &= H[N^i \partial_i \underline{N}] \\
\{H(\underline{N}), G_a(\theta^a)\} &= 0; \\
\big[H(\underline{N}), H(\underline{M})\big] &= H_i[\big(\underline{N} \partial_j \underline{M} - \underline{M} \partial_j \underline{N}\big) H^{ij}].
\end{aligned} \tag{3}$$

Equation (3) is a first class system due to closure of the algebra, and is therefore consistent in the Dirac sense. The kinematic constraints H_i and G_a, which generate spatial diffeomorphisms and $SU(2)_-$ gauge transformations, form a closed six dimensional subalgebra of their own $\boldsymbol{A}_{Kin} = \boldsymbol{A}_{gauge} \times \boldsymbol{A}_{diff}$. For a theory invariant under $SU(2)_-$ gauge transformations and spatial diffeomorphisms, this would constitute a first class system.[3] However, as noted for (1), the Hamiltonian constraint H by itself in the Ashtekar variables is a second class constraint, which still necessitates its enlargement to include the kinematic constraints for closure.

Clearly it is not possible in the full theory either in metric or in Ashtekar variables to consistently implement the kinematic constraints, leaving behind a reduced phase space of dynamics generated solely by the Hamiltonian constraint.[4] Since the Hamiltonian constraint appears to be intractable in the full theory, it is problematic to eliminate it via Dirac brackets.[5] It would be fortuitous if the roles of H and (H_i, G_a) in (3) could be reversed, such that H forms its own first class system independently of $\boldsymbol{A}_{Kin}$. Since the kinematic constraints are certainly tractable, then they could then be eliminated via Dirac brackets and one would be left with the Hamiltonian constraint subalgebra, which governs the physical evolution of the theory. The obstruction to the implementation of this idea, as is clear from (1) and (3), is that the commutator of two Hamiltonian constraints does not produce a Hamiltonian constraint. We see that this obstruction becomes eliminated

[3]This could arise for example from the Viquar Hussain model, which essentially is general relativity in Ashtekar variables with the Hamiltonian constraint missing while the kinematic constraints remain intact [4], [5].

[4]This is possible in certain minisuperspace models where the variables are spatially homogeneous, whence the derivatives acting on the lapse functions vanish, allowing two Hamiltonian constraints to strongly commute. But this situation is in a sense a trivialization of the diffeomorphism group, which is incongruous with the full theory.

[5]In the loop representation of quantum gravity, it is possible to construct a representation of $\boldsymbol{A}_{Kin}$ using gauge invariant spin network states and group averaging techniques. Such states satisfy the kinematic constraints by construction, however due not lie in the kernel the Hamiltonian constraint in the full theory. We believe that this failure can be traced to the observation that H does not form its own subalgebra as noted above.

when one expresses general relativity using the instanton representation (see e.g. Paper II with references therein).

The organization of this paper is as follows. In sections 2 and 3 we derive the instanton representation using the Ashtekar variables as a starting point, and create a library of preliminary results needed for computing the Poisson brackets in this representation. It is necessary to introduce some new terminology, to allow for the fact that the symplectic two form in the instanton representation contains functional dependence on the dynamical variables. In section 4 we compute the Possion algebra of constraints and we analyse the results and their physical interpretation in section 5.

2 From Ashtekar variables into the instanton representation

There is some brief terminology which we must introduce prior to proceeding with the present paper. The phase space Ω for a physical system is said to admit a $(p,q)_{HH}$ structure if there exist coordinates where the phase space variables (p,q) constitute a globally canonical pair of $(momentum, configuration)$ variables and a globally homogenenous symplectic two form $\boldsymbol{\Omega}$. A globally homogeneous symplectic two form $\boldsymbol{\Omega}$ is a two form which in the phase space coordinates (p,q) globally takes on the canonical form $\boldsymbol{\Omega}=\delta p\wedge\delta q\ \forall(p,q)\in\Omega$. We will examine the constraints algebra of general relativity induced by two main structures, $(p,q)_{HH}$ and $(p,q)_{NH}$, the substript N meaning that while the symplectic two form is globally homogeneous, the coordinates on the phase space are not holonomic.[6]

The Ashtekar formulation of general relativity is an example of a $(p,q)_{HH}$ structure. The 3+1 decomposition of the action for general relativity in this formulation is [2], [3] yields a totally constrained system

$$I_{Ash}=\int dt\int_{\Sigma}d^3x\Big(\widetilde{\sigma}^i_a\dot{A}^a_i-\epsilon_{ijk}N^i\widetilde{\sigma}^j_aB^k_a+A^a_0D_i\widetilde{\sigma}^i_a \\ -\underline{N}\big(\frac{\Lambda}{3}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c+\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_bB^k_c\big)\Big). \tag{4}$$

The initial value constraints are respectively the diffeomorphism H_i, Gauss' law G_a and the Hamiltonian constraints H, which are smeared by their respective Lagrange multipliers the shift function N^i, gauge angle A^a_0, and lapse density function $\underline{N}=N(\text{det}\widetilde{\sigma})^{-1/2}$. For Lorentz signature we can take N to be imaginary. The basic phase space variables are a self-dual $SU(2)_-$ connection and the densitized triad $\Omega_{Ash}=(A^a_i,\widetilde{\sigma}^i_a)$, which form a canonical pair with respect to a $(p,q)_{HH}$ structure as defined above since they are globally holonomic. The $(A^a_i,\widetilde{\sigma}^i_a)_{HH}$ structure of the Ashtekar variables implies the following Poisson brackets induced by the canonical pair $(\widetilde{\sigma}^i_a,A^a_i)$ on phase space functions f and g

$$\{f,g\}=\int_{\Sigma}d^3z\Big(\frac{\delta f}{\delta\widetilde{\sigma}^i_a(z)}\frac{\delta g}{\delta A^a_i(z)}-\frac{\delta g}{\delta\widetilde{\sigma}^i_a(z)}\frac{\delta f}{\delta A^a_i(z)}\Big). \tag{5}$$

When f and g are taken to be the initial value constraints of general relativity, then (5) produces the algebra (3) which is first class.

There remain at least two main open issues for the full theory implied by (4). First, it remains to be determined the projection from the full

[6]There seems to be no difference mathematically between a $(p,q)_{NH}$ and a $(p,q)_{HN}$ structure as far as the dynamics are concerned. The above notation is introduced in Paper 8 of this series.

unconstrained phase space Ω_{Ash} to the reduced phase space Ω_{Phys}, where the algebra (3) is Dirac consistent. Secondly, the quantum Hamiltonian dynamics need to be consistently implemented, for the full theory, with respect to this projection.[7]

To obtain the instanton representation of Plebanski gravity, define a complex matrix $\Psi_{ae} \in SU(2)_- \otimes SU(2)_-$ such that the following relation holds

$$\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e, \tag{6}$$

where B^i_e is the magnetic field derived from the spatial part of the curvature of the Ashtekar connection $B^i_e = \frac{1}{2}\epsilon^{ijk} F^e_{jk}[A]$. Equation (6) is known as the CDJ Ansatz, which holds when Ψ_{ae} and B^i_e are nondegenerate three by three matrices. Next, substitute (6) into the action (4), at the level of its 3+1 decomposition. This is given by

$$I_{Inst} = \frac{1}{G}\int^T_0 \int_\Sigma d^3x \Big[\Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 \mathbf{w}_e\{\Psi_{ae}\} - \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} \\ + N(\det B)^{1/2}(\det\Psi)^{1/2}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big], \tag{7}$$

where we have defined $\mathbf{w}_e = B^i_e D_i$, with D_i the $SU(2)_-$ covariant derivative, given by

$$D_i\{\Psi_{ae}\} = \partial_i \Psi_{ae} + A^b_i \big(f_{abf}\Psi_{ge} + f_{ebg}\Psi_{ag}\big) \tag{8}$$

in the tensor representation.[8] If (6) were a canonical transformation, then the phase space structure of (7) would imply that the variable canonically conjugate to Ψ_{ae} is an object X^{ae} whose time derivative is $B^i_e \dot{A}^a_i$. However, (6) is not a canonical transformation, which can be seen as follows. The symplectic two form on Ω_{Ash} is given by

$$\mathbf{\Omega}_{Ash} = \int_\Sigma d^3x \delta\widetilde{\sigma}^i_a(x) \wedge \delta A^a_i(x) = \delta\Big(\int_\Sigma d^3x \widetilde{\sigma}^i_a(x)\delta A^a_i(x)\Big) = \delta\boldsymbol{\theta}_{Ash}, \tag{9}$$

which is the exterior derivative of the canonical one form $\boldsymbol{\theta}_{Ash}$. Using the functional Liebniz rule in conjuction with the variation of (6) we have $\delta\widetilde{\sigma}^i_a = B^i_e \delta\Psi_{ae} + \Psi_{ae}\delta B^i_e$, which transforms the left hand side of (9) into

[7] It is the aim of this series of papers to address these issues using the instanton representation of Plebanski gravity, the present paper treating the classical constraints algebra.

[8] It will be convenient for the purposes of the Gauss' law constraint to define a new term, 'magnetic helicity density matrix' $C_{be} = A^b_i B^i_e$, which is the matrix product of the Ashtekar connection with the magnetic field with spatial indices contracted to produce a $SU(2)_- \otimes SU(2)_-$-valued matrix.

$$\mathbf{\Omega}_{Inst} = \int_\Sigma d^3x \delta\Psi_{ae} \wedge B^i_e \delta A^a_i + \int_\Sigma \Psi_{ae} \delta B^i_e \wedge \delta A^a_i. \tag{10}$$

It happens that the Soo one-forms are given by $\delta X^{ae} = B^i_e \delta A^a_i$. The variables X^{ae} were first identified by Chopin Soo in [6] and [7] for their natural adaptability to the gauge invariances of general relativity in the Ashtekar variables.[9] In terms of the Soo potentials, (10) reduces to

$$\omega = \int_\Sigma d^3x \delta\Psi_{ae} \wedge \delta X^{ae} + \int_\Sigma \epsilon_{ijk} \Psi_{ae} \delta(D_j A^e_k) \wedge \delta A^a_i. \tag{11}$$

The first term of (11) is the symplectic two-form corresponding to (X^{ae}, Ψ_{ae}) seen as a canonical pair. However, the second term in general does not vanish and it is not clear how to write it explicitly in terms of X^{ae}. Therefore, the transformation $(A^a_i, \widetilde{\sigma}^i_a) \rightarrow (X^{ae}, \Psi_{ae})$ is in general noncanonical. It is at this point where we must introduce a new postulate or principle.

Let us postulate the action (7) as the starting point for a new description of gravity, where (X^{ae}, Ψ_{ae}) define a $(p,q)_{HN}$ structure. By this we mean that while the coordinates X^{ae} do not exist, they may still be used to compute Poisson brackets since only the variations δX^{ae}, which are globally well-defined, are required. Then the Ashtekar formulation becomes derived from (7) via the substitution (6), for nondegenerate configurations. It may be helpful for the reader to think of the instanton representation as corresponding to a phase space $\Omega_{Inst} = (\Psi_{ae}, A^a_i)_{HN}$ where the Poisson brackets are given by

$$\{f, g\} = \int_\Sigma d^3z \Big(\frac{\delta f}{\delta \Psi_{bf}(z)} (B^{-1}(z))^f_j \frac{\delta g}{\delta A^b_j(z)} - \frac{\delta g}{\delta \Psi_{bf}(z)} (B^{-1}(z))^f_j \frac{\delta f}{\delta X^{bf}(z)} \Big). \tag{12}$$

The symplectic two form contains the Ashtekar magnetic field B^i_a as part of its structure, which must be globally nondegenerate on Ω_{Inst} in order to exist.[10]

[9] While X^{ae} do not in general exist as holonomic coordinates on configuration space Γ, their variations $\delta X^{ae} \in T^*_X(\Gamma)$ live in the cotangent space to Γ, which is globally well-defined. Therefore, we will make use of δX^{ae}, and never actually X^{ae} itself, in the computation of Poisson brackets.

[10] The nondegeneracy of B^i_a is a necessary condition for (6) to be valid, hence we will restrict ourself to these configurations in this paper.

3 Ingredients for the Poisson algebra in the instanton representation

We have changed the description of general relativity into a new set of coordinates $\Omega_{Inst} = (\Psi_{ae}, A^a_i)_{HN}$ (or alternatively, nonholonomic coordinates $(\Psi_{ae}, X^{ae})_{NH}$) where (6) must be read from right to left in order to obtain the original Ashtekar variables. There are some consistency checks which must be made to verify that (6) is an allowed transformation. One consistency check, which will be the main focus of the present paper, is for closure of the constraints algebra on Ω_{Inst}, given that the algebra closes on Ω_{Ash}. The fundamental Poisson brackets on Ω_{Ash} are given by

$$\{A^a_i(x), \widetilde{\sigma}^j_b(y)\}_{HH} = \delta^a_b \delta^j_i \delta^{(3)}(\mathbf{x}, \mathbf{y}), \tag{13}$$

which signifies the $(\widetilde{\sigma}^i_a, A^a_i)_{HH}$ structure, as evidenced by the fact that the right hand side is independent of the phase space variables. Additionally, one has the vanishing Poisson brackets

$$\{A^a_i(x), A^j_b(y)\} = \{\widetilde{\sigma}^i_a(x), \widetilde{\sigma}^j_b(y)\} = 0. \tag{14}$$

Equation (13) implies the following Schrödinger representation

$$\widetilde{\sigma}^i_a(x) \longrightarrow \frac{\delta}{\delta A^a_i(x)} \in T_A(\Gamma_{Ash}), \tag{15}$$

which defines vector fields $\delta/\delta A^a_i \in T_A(\Gamma_{Ash})$ in the tangent space to the Ashtekar configuration space. Dual to $T_A(\Omega_{Ash})$ is the cotangent space $T^*_A(\Omega_{Ash})$, which is spanned by functional one forms δA^a_i such that

$$\left\langle \frac{\delta}{\delta A^a_i(x)} \middle| \delta A^b_j(y) \right\rangle = \delta^b_a \delta^i_j \delta^{(3)}(\mathbf{x}, \mathbf{y}). \tag{16}$$

Let us now examine the situation for the instanton representation. Substituting (6) into (13), we obtain

$$\{A^a_i(x), \Psi_{be}(y) B^j_e(y)\} = \delta^a_b \delta^j_i \delta^{(3)}(\mathbf{x}, \mathbf{y}). \tag{17}$$

Applying the Liebniz rule to (17), we obtain

$$\{A^a_i(x), \Psi_{be}(y)\} B^j_e(y) + \Psi_{be}(y) \{A^a_i(x), B^j_e(y)\} = \delta^a_b \delta^j_i \delta^{(3)}(\mathbf{x}, \mathbf{y}). \tag{18}$$

Using the first equation of (14), one sees that the second term on the left hand side of (18) vanishes. Transferring the magnetic field to the right hand side we then obtain

$$\{A^a_i(x),\Psi_{be}(y)\}_{HN}=\delta^a_b(B^{-1})^e_j\delta^{(3)}(\mathbf{x},\mathbf{y}). \tag{19}$$

The notation should hopefully be clear, in that in order to retain globally holonomic coordinates in the new variables (6), it was necessary to deform the canonical structure such that the right hand side of the commutation relations contains field dependence. Observe that one may from (19) read off the following Schrödinger representation in analogy to (15)

$$\Psi_{be}(x)\longrightarrow(B^{-1}(x))^e_j\frac{\delta}{\delta A^b_j(x)}\equiv\frac{\delta}{\delta X^{be}(x)}\in T_X(\Gamma_{Inst}), \tag{20}$$

which defines functional one forms

$$B^i_e(x)\delta A^a_i(x)\equiv\delta X^{ae}\in T^*_X(\Gamma_{Inst}) \tag{21}$$

such that

$$\Big\langle\frac{\delta}{\delta X^{ae}(x)}\Big|\delta X^{bf}(y)\Big\rangle=\delta^b_a\delta^f_e\delta^{(3)}(\mathbf{x},\mathbf{y}). \tag{22}$$

Note, while X^{ae} cannot be defined on account of the fact that $\delta X^{ae}=B^i_e\delta A^a_i$ is not in general an exact one form, (19), (20), (21) and (22) are still well-defined. We will now prove the converse of the previous steps. Namely, we will show even if X^{ae} is not globally defined on Ω_{Inst}, that this does not preclude the formulation of the following elementary Poisson brackets

$$\{X^{bf}(\mathbf{x},t),\Psi_{ae}(\mathbf{y},t)\}_{NH}=\beta G\delta^a_b\delta^e_f\delta^{(3)}(\mathbf{x},\mathbf{y}), \tag{23}$$

with trivial brackets

$$\{X^{ae}(\mathbf{x},t),X^{bf}(\mathbf{y},t)\}=\{\Psi_{ae}(\mathbf{x},t),\Psi_{bf}(\mathbf{y},t)\}=0, \tag{24}$$

where we choose $\beta=i$ for Lorentzian signature spacetimes.[11]

To show this, we first write (6) in reverse

[11]The quantization of GR using (X^{ae},Ψ_{ae}) is the topic of a different paper within the instanton representation series.

$$\Psi_{ae} = \widetilde{\sigma}^i_a (B^{-1})^i_e \tag{25}$$

and subsitute into (23). This implies that

$$\{X^{bf}(y), (B^{-1}(x))^e_i \widetilde{\sigma}^i_a(x)\} = \{X^{bf}(y), (B^{-1}(x))^e_i\} \widetilde{\sigma}^i_a(x)$$
$$+ (B^{-1}(x))^e_i \{X^{bf}(y), \widetilde{\sigma}^i_a(x)\} = \delta^b_a \delta^f_e \delta^{(3)}(\mathbf{x}, \mathbf{y}). \tag{26}$$

Assuming that X^{ae}, even if it may not exist globally on configuration space Γ_{Inst}, is not part of the momentum space P_{Inst}, then the first term on the right hand side of (26) must vanish yielding the relation

$$(B^{-1}(x))^e_i \{X^{bf}(y), \widetilde{\sigma}^i_a(x)\} = \delta^b_a \delta^f_e \delta^{(3)}(\mathbf{x}, \mathbf{y}). \tag{27}$$

Transferring the magnetic field to the right hand side yields the relation

$$\{X^{bf}(y), \widetilde{\sigma}^i_a(x)\} = \delta^b_a B^f_i(x) \delta^{(3)}(\mathbf{x}, \mathbf{y}), \tag{28}$$

which implies in the functional Schrödinger representation that

$$\widetilde{\sigma}^i_a(x) \rightarrow \frac{\delta}{\delta A^a_i} = B^i_e(x) \frac{\delta}{\delta X^{ae}(x)}. \tag{29}$$

Left-multiplying (29) by the inverse of the magentic field, which is nondegenerate due to (25), we obtain

$$\frac{\delta}{\delta X^{af}(x)} = (B^{-1}(x))^f_i \frac{\delta}{\delta A^a_i(x)} \in T_X(\Gamma_{Inst}), \tag{30}$$

which is the same as (20). So we see that although the transformation (6) is noncanonical, the ingredients necessary for computing functional variations in the new set of variables are still perfectly well-defined.

3.1 A few preliminary results

We will compute the Poisson algebra of constraints, using the initial value constraints from (7). The initial value constraints can be written explicitly in terms of the holonomic coordinates (Ψ_{ae}, A^b_j). These are the diffeomorphim and Gauss' law constraints

$$\vec{H}[\vec{N}] = \int_\Sigma d^3x \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae}; \quad \vec{G}[\vec{\theta}] = \int_\Sigma d^3x \theta^a \mathbf{w}_e \{\Psi_{ae}\}, \tag{31}$$

and the Hamiltonian constraint

$$H[N]=\int_\Sigma d^3xN(\mathrm{det}B)^{1/2}\sqrt{\mathrm{det}\Psi}\big(\Lambda+\mathrm{tr}\Psi^{-1}\big). \tag{32}$$

In the computation of the constraints algebra on Ω_{Inst} we will need to make repeated use of the following result, which evaluates

$$I=\frac{\delta}{\delta A^e_l(x)}\int_\Sigma d^3yB^i_a(y)F(y) \tag{33}$$

for an arbitrary function F smeared with the Ashtekar magnetic field B^i_a. This is given in terms of the constituent fields by

$$\frac{\delta}{\delta A^e_l(x)}\int_\Sigma d^3yF(y)\Big(\epsilon^{ijk}\partial_jA^a_k(y)+\frac{1}{2}\epsilon^{ijk}f^{abc}A^b_j(y)A^c_k(y)\Big). \tag{34}$$

We now integrate (34) by parts, discarding boundary terms,[12] obtaining

$$\begin{aligned}I=-\epsilon^{ijk}\int_\Sigma d^3y(\partial_jF(y))\frac{\delta A^a_k(y)}{\delta A^e_l(x)}+\epsilon^{ijk}f^{abc}\int_\Sigma d^3yA^b_j(y)\frac{\delta A^c_k(y)}{\delta A^e_l(x)}\\=-\epsilon^{ijk}\int_\Sigma d^3y\delta^a_e\delta^l_k\delta^{(3)}(\mathbf{x},\mathbf{y})+\epsilon^{ijk}f^{abc}\int_\Sigma d^3yA^b_j(y)\delta^c_e\delta^l_k\delta^{(3)}(\mathbf{x},\mathbf{y})\\=-\delta_{ae}\epsilon^{ijl}\partial_jF+\epsilon^{ijl}f^{abe}A^b_jF=\epsilon^{ijl}\big(-\delta_{ae}\partial_j+f_{abe}A^b_j\big)F.\end{aligned} \tag{35}$$

We now define the following notation

$$\overline{D}^{il}_{ae}F=\epsilon^{ijl}\big(-\delta_{ae}\partial_j+f_{abe}A^b_j\big)F;\quad D^{il}_{ae}F=\epsilon^{ijl}\big(\delta_{ae}\partial_j+f_{abe}A^b_j\big)F \tag{36}$$

so that for two functions F and G with sufficiently rapid fall-off conditions,

$$\int_\Sigma d^3xF(\overline{D}^{ij}_{ab}G)=\int_\Sigma d^3xG(D^{ij}_{ab}F). \tag{37}$$

Hence under integrals involving $SU(2)_-$ vectors, $\overline{D}^{ij}_{ab}$ can be converted into a covariant derivative operator D^{ij}_{ab}

$$\int_\Sigma d^3x\overline{D}^{ij}_{ab}F_b=\int_\Sigma\epsilon^{ijk}(D_k)_{ab}F_b. \tag{38}$$

[12]We have assumed either a spatial 3-manifold Σ without boundary, or sufficiently rapid falloff of the fields on the boundary $\partial\Sigma$. This enables us to exploit to the maximum extent the smooth structure of the instanton representation variables at the canonical level.

We will now develop a library of the ingredients necessary to the Poisson algebra with respect to the $(\Psi_{ae}, A^b_j)_{HN}$ structure, using the initial value constraints from (31) and (32). Starting with the smeared diffeomorphism constraint H_i, the variation with respect to Ψ_{bf} is given by

$$\frac{\delta \vec{H}[\vec{N}]}{\delta \Psi_{bf}} = \epsilon_{ijk} N^i B^j_b B^k_f = (\vec{N} \times \vec{B})_{kb} B^k_f \tag{39}$$

which is antisymmetric in indices b, f. For (42) we have defined the following notation regarding cross products with a magnetic field

$$(\vec{N} \times \vec{B})_{ia} = \epsilon_{ijk} N^j B^k_a; \quad (\vec{\theta} \times \vec{B})^j_a \equiv f_{acd} \theta^c B^j_d. \tag{40}$$

The first equation of (40) is the cross product of a spatial 3-vector $\vec{N} \equiv N^i$ with $\vec{B}_a \equiv B^i_a$, seen as a triple of spatial 3-vectors labelled by the internal index a. The second equation of (40) is the cross product of an internal 3-vector $\vec{\theta} \equiv \theta^a$ with $\vec{B}^i \equiv B^i_a$, seen as a triple of internal vectors labelled by the spatial component i.[13]

To compute variations with respect to X^{bf} we will use the result that $\delta/\delta X^{bf} = (B^{-1})^f_j \delta/\delta A^b_j$ from (30), since the coordinate X^{bf} is not defined on Γ_{Inst}. Integrating by parts, we obtain

$$\begin{aligned} \frac{\delta \vec{H}[\vec{N}]}{\delta X^{bf}(x)} &= 2(B^{-1})^f_l \int_\Sigma d^3x' N^i \epsilon_{ijk} \frac{\delta}{\delta A^b_l} (B^j_a B^k_e) \Psi_{[ae]} \\ &= 2(B^{-1})^f_l \epsilon_{ijk} \Big[\overline{D}^{jl}_{ab} (N^i B^k_e \Psi_{[ae]}) + \overline{D}^{kl}_{eb} (N^i B^j_a \Psi_{[ae]}) \Big] \\ &= 4\epsilon_{ijk} (B^{-1})^f_l \overline{D}^{jl}_{ab} (N^i B^k_e \Psi_{[ae]}) \equiv -4 (B^{-1})^f_l \overline{D}^{jl}_{ab} ((\vec{N} \times \vec{B})_{je} \Psi_{[ae]}) \end{aligned} \tag{41}$$

where we have relabelled indices on the second term and used the antisymmetry of ϵ_{ijk}. Note that we have left $(B^{-1})^f_l = (B^{-1}(x))^f_j$ outside the integral, since it is independent of the dummy variable of integration x'. The spatial gradients $\frac{\partial}{\partial x'^i}$ originating from $B^i_e = B^i_e(x')$ then act on all quantities remaining within the integrand which depends only on x'.

Moving on to the variational derivatives of the smeared Gauss' law constraint with respect to Ψ_{ae},

$$\begin{aligned} \frac{\delta G_a[\theta^a]}{\delta \Psi_{ch}(x)} &= \frac{\delta}{\delta \Psi_{ch}(x)} \int_\Sigma d^3y \theta^a(y) \Big(\delta^{af} B^i_g(y) \frac{\partial}{\partial y^i} + B^i_e(y) A^b_i(y) f_{abfge} \Big) \Psi_{fg}(y) \\ &= \int_\Sigma d^3y \Big(-\delta^{af} \frac{\partial}{\partial y^i} (\theta^a(y) B^i_g(y)) + \theta^a(y) B^i_e(y) A^b_i(y) f_{abfge} \Big) \delta^c_f \delta^h_g \delta^{(3)}(\mathbf{x}, \mathbf{y}) \end{aligned} \tag{42}$$

[13] The notation should hopefully be unambiguous in that in the latter case the spatial index occurs in a raised position, while in a lowered position in the former case.

where $f_{abfge} = f_{abf}\delta_{ge} + f_{ebg}\delta_{af}$. Further simplification of (42) yields

$$
\begin{aligned}
&-\delta^{ac}\partial_i(\theta^a B^i_h) + \theta^a C_{be}\big(f_{abc}\delta_{he} + f_{ebh}\delta_{ac}\big) \\
&= -\delta^{ac}B^i_h\partial_i\theta^a - \delta^{ac}\partial_i B^i_h + \theta^a C_{be}\big(f_{abc}\delta_{he} + f_{ebh}\delta_{ac}\big) = -B^i_h D_i\theta^c, \qquad (43)
\end{aligned}
$$

where the second and fourth terms have cancelled due to the gauge Bianchi identity $D_iB^i_a = \partial_i B^i_a + f_{abc}A^b_i B^i_c = 0$. The result is the covariant derivative acting on the angles θ^c. Moving on to the configuration variables, we have

$$
\begin{aligned}
\frac{\delta G_a[\theta^a]}{\delta X^{bf}(x)} &= (B^{-1})^f_l \frac{\delta}{\delta A^b_l}\int_\Sigma d^3y\theta^a B^i_e\Big(\partial_i\Psi_{ae} + \big(f_{adh}\delta_{ge} + f_{edg}\delta_{ah}\big)A^d_i\Psi_{hg}\Big) \\
&= (B^{-1})^f_l\overline{D}^{il}_{eb}(\theta^a D_i\Psi_{ae}) + \theta^a\big(f_{abh}\delta_{gf} + f_{fbg}\delta_{ah}\big)\Psi_{hg} \equiv \overline{W}^{hg}_{bf}(\vec{\theta})\Psi_{hg}. (44)
\end{aligned}
$$

In (44) the functional derivative acted on the pre-factor B^i_e, which induced an integration by parts, and also on the factor A^d_i in brackets.[14] For the smeared Hamiltonian constraint we have

$$
\begin{aligned}
\frac{\delta H[N]}{\delta X^{bf}(x)} &= (B^{-1})^f_l\int_\Sigma d^3yN'(\mathrm{det}\Psi)^{1/2}(\Lambda + \mathrm{tr}\Psi^{-1})\frac{\delta}{\delta A^b_l}(\mathrm{det}B)^{1/2} \\
&= \frac{1}{2}(B^{-1})^f_l\overline{D}^{kl}_{ab}\Big(N(B^{-1})^a_k(\mathrm{det}B)^{1/2}(\mathrm{det}\Psi)^{1/2}(\Lambda + \mathrm{tr}\Psi^{-1})\Big) \\
&= \frac{1}{2}(B^{-1})^f_l\overline{D}^{kl}_{ab}\big((B^{-1})^a_kNH\big). \qquad (45)
\end{aligned}
$$

Equation (45) contains a part proportional to H itself, which should weakly vanish on the constraint surface. Note that there is also a contribution proportional to the spatial gradient given by

$$
N(B^{-1})^f_l(B^{-1})^a_k\epsilon^{klm}\delta_{ab}\partial_m H = N(\mathrm{det}B)^{-1}\epsilon^{bfg}B^m_g\partial_m H = N(\mathrm{det}B)^{-1}\epsilon^{bfg}\mathbf{v}_g\{H\} (46)
$$

where $\mathbf{v}_g = B^i_g\partial_i$ is a triple of vector fields constructed from the magnetic field B^i_g, labelled by the internal index g. Note that the symmetric part of this contribution vanishes.

Moving on to the variational derivative with respect to Ψ_{bf}, we have

$$
\begin{aligned}
\frac{\delta H[N]}{\delta\Psi_{bf}(x)} &= \frac{\delta}{\delta\Psi_{bf}(x)}\int_\Sigma d^3yN(\mathrm{det}B)^{1/2}(\mathrm{det}\Psi)^{1/2}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big) \\
&= N(\mathrm{det}B)^{1/2}(\mathrm{det}\Psi)^{1/2}\Big[\frac{1}{2}(\Psi^{-1})^{bf}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big) - (\Psi^{-1}\Psi^{-1})^{bf}\Big] \\
&= N\big(\frac{1}{2}(\Psi^{-1})^{bf}H - (\mathrm{det}B)^{1/2}\eta^{bf}\big) = N(\mathrm{det}B)^{1/2}\boldsymbol{M}^{bf}. \qquad (47)
\end{aligned}
$$

[14]Note in (44) that there are two derivatives which act on Ψ_{ae}, which defines $\overline{W}^{ae}_{bf}$. The unbarred version W^{hg}_{bf} will denote the counterpart to $\overline{W}^{hg}_{bf}$ after integration by parts has been carried out to transfer the action of the outside spatial gradient away, leaving behing a single spatial derivative acting on Ψ_{ae}.

Equation (47) has a contribution proportional to the Hamiltonian constraint H, and a contribution due to a quantity η^{bf}, given by

$$\eta^{bf} = \sqrt{\mathrm{det}\Psi}(\Psi^{-1}\Psi^{-1})^{bf}, \tag{48}$$

where $\mathrm{det}\Psi \neq 0$. Equation (74) is not to be confused with the Minkoski metric, since it uses internal indices and also depends on position. The quantity η^{bf} will take on the interpretation of an internal $SU(2)_- \otimes SU(2)_-$ metric related to the 3-metric h_{ij} induced on 3-space Σ from the spacetime metric $g_{\mu\nu}$. The 3-metric can be obtained from η^{bf} via the relation

$$h_{ij} = (\mathrm{det}\eta)^{-1}\eta^{bf}(B^{-1})^b_i(B^{-1})^f_j(\mathrm{det}B). \tag{49}$$

.

4 Classical algebra of constraints in the instanton representation

4.1 A few preliminaries on notation

To put the transformations generated by the kinematic constraints into perspective, let us consider first the effect of a gauge transformation on the connection. In the original Ashtekar variables this is given by

$$\delta_{\vec{\theta}} A^a_i = \{A^a_i, D_i \widetilde{\sigma}^i_a\} = -D_i \theta^a, \tag{50}$$

which in the nonholonomic coordinates is given by

$$\delta_{\vec{\theta}} X^{ae} = B^i_e \delta_{\vec{\theta}} A^a_i = -B^i_e D_i \theta^a \equiv -\mathbf{w}_e\{\theta^a\} = \frac{\delta \vec{G}[\vec{\theta}]}{\delta \Psi^{ae}}. \tag{51}$$

A spatial diffeomorphism of the connection is given by its Lie derivative

$$L_{\vec{N}} A^a_i = (\partial_i N^j) A^a_j + N^j \partial_j A^a_i = D_i(N^j A^a_j) - N^j F^a_{ij}. \tag{52}$$

Multiplication of (52) by the magnetic field yields

$$B^i_e L_{\vec{N}} A^a_i = B^i_e D_i(N^j A^a_j) - B^i_e(\vec{N} \times \vec{B})_{ia}, \tag{53}$$

which can be written in nonholonomic coordinates as

$$L_{\vec{N}} X^{ae} = \mathbf{w}_e(N^j A^a_j) - \frac{\delta \vec{H}[\vec{N}]}{\delta \Psi_{ae}} \tag{54}$$

where $\mathbf{w}_e = B^i_e D_i$. Let us now write the constraints in the instanton representation in standard smeared form. The diffeomorphism constraint is given by

$$H_i = \epsilon_{ijk} B^j_a B^k_e \Psi_{ae} = (\vec{N} \times \vec{B})_{ka} B^k_e \Psi_{ae}, \tag{55}$$

which is distinguished by the fact that it is linear in the antisymmetric part of Ψ_{ae}. In smeared form this is given by

$$\begin{aligned} \vec{H}[\vec{N}] = \Psi_{[ae]}[V^{ae}(\vec{N})] &= \int_\Sigma d^3x \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{[ae]}(x) \\ &= \int_\Sigma d^3x \big(\delta_{\vec{N}\cdot A} X^{ae} - L_{\vec{N}} X^{ae}\big) \Psi_{[ae]}, \end{aligned} \tag{56}$$

which upon comparison with (53) and (54) shows that the antisymmetric $\Psi_{[ae]}$ can be regarded as the generator of a spatial diffeomorphism adjusted by a gauge transformation with field-dependent parameter $N^j A^a_j$.

The Gauss' Law constraint is given by

$$G_a = \mathbf{v}_e\{\Psi_{ae}\} + C^{fg}_a \Psi_{fg} \equiv \mathbf{w}_e\{\Psi_{ae}\} \tag{57}$$

and is distinguished by two structures. First there is a triple of vector fields $\mathbf{v}_a = B^i_a \partial_i$ constructed from the $SO(3,C)$ magnetic field which contracts one of the indices on Ψ_{ae} as a kind of internal divergence operator. The second structure is an object

$$C^{fg}_a = \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big)C_{be}, \tag{58}$$

where $C_{be} = A^b_i B^i_e$ is defined as the 'magnetic helicity density matrix'. The effect of (58) in (57) is to act on Ψ_{ae}, seen as a second-rank $SO(3,C)$-valued tensor, in the tensor representation of the gauge group. Both structures in (57) will in general be smeared with gauge parameters, and as in the case of single parameters will combine to act on Ψ_{ae} in some kind of tensor representation (8). This is given by[15]

$$\vec{G}[\vec{\theta}] = W^{ae}(\vec{\theta})\{\Psi_{ae}\} = \int_\Sigma d^3x \Big(\mathbf{v}^{fg}(\vec{\theta}) + \theta^a C^{fg}_a\Big)\Psi_{fg} = -\int_\Sigma d^3x (\delta_{\vec{\theta}} X^{ae})\Psi_{ae}, \tag{59}$$

where we have defined a parameter-dependent vector field $\mathbf{v}^{fg}(\vec{\theta}) \equiv \theta^f \mathbf{v}_g$. Comparison of (59) with (51) shows that Ψ_{ae} can be regarded as the generator of a gauge transformation.

Lastly, the Hamiltonian constraint is given by

$$H = (\mathrm{det}B)^{1/2}\sqrt{\mathrm{det}\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \tag{60}$$

Note, due to the nondegeneracy of B^i_a and Ψ_{ae} and to $SO(3,C)$ invariance, that (60) can be written in the equivalent form

$$\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \sim 0, \tag{61}$$

[15] As a convention, we will use $V^{ae}(\vec{N})$ and $W^{ae}(\vec{\theta})$ to signify the smearing functions corresponding to the diffeomorphism and Gauss' law constraints repectively. The boldface versions of these $\boldsymbol{V}^{ae}(\vec{N})$ and $\boldsymbol{W}^{ae}(\vec{\theta})$ will signify the same constraints, but containing momentum-dependent structure functions.

where λ_1, λ_2 and λ_3 are the eigenvalues of Ψ_{ae}. As a note of caution, the solutions of the initial value constraints can only be used subsequent to, and not before, computing the algebra of constraints.

We shall compute the constraints algebra using Poisson brackets

$$\{f,g\}_{NH} = \int_\Sigma d^3x \Big(\frac{\delta f}{\delta \Psi_{bf}} \frac{\delta g}{\delta X^{bf}} - \frac{\delta g}{\delta \Psi_{bf}} \frac{\delta f}{\delta X^{bf}} \Big)$$
$$\equiv \int_\Sigma d^3x \Big(\frac{\delta f}{\delta \Psi_{bf}} (B^{-1})^f_j \frac{\delta g}{\delta A^b_j} - \frac{\delta g}{\delta \Psi_{bf}} (B^{-1})^f_j \frac{\delta f}{\delta A^b_j} \Big) = \{f,g\}_{HN}. \qquad (62)$$

We have written (62) to remind the reader that using holonomic coordinates with a field-dependent symplectic structure is mathematically the same as using nonholonomic with a field-independent canonical structure, when computing Poisson brackets. As a review prior to proceeding, let us quote the results regarding variational derivatives, rewritten in bf indices.

$$\frac{\delta \vec{H}[\vec{N}]}{\delta \Psi_{bf}} = (\vec{N} \times \vec{B})_{kb} B^k_f; \quad \frac{\delta \vec{H}[\vec{N}]}{\delta X^{bf}} = -4(B^{-1})^f_l \overline{D}^{jl}_{ab}((\vec{N} \times \vec{B})_{je} \Psi_{[ae]});$$
$$\frac{\delta \vec{G}[\vec{\theta}]}{\delta \Psi_{bf}} = -B^i_f D_i \theta^b \equiv \mathbf{w}_f\{\theta^b\}; \quad \frac{\delta \vec{G}[\vec{\theta}]}{\delta X^{bf}} = \overline{W}^{ae}_{bf}[\vec{\theta}] \Psi_{ae};$$
$$\frac{\delta H[N]}{\delta \Psi_{bf}} = (\det B)^{1/2} \boldsymbol{M}^{bf} N; \quad \frac{\delta H[N]}{\delta X^{bf}} = \frac{1}{2} (B^{-1})^f_l \overline{D}^{kl}_{ab}((B^{-1})^a_k HN). \quad (63)$$

Numerical pre-factors will not be important in computing the algebra, and we will occasionally omit them. Now that we have defined all notations and conventions, we are ready to compute the algebra of constraints on the phase space Ω_{Inst}.

4.2 Computation of the constraint subalgebras

We will now compute the algebra of constraints, starting with constraints of the same type. In what follows we will omit the HN and NH notation, since it should be clear form the context. Starting with the Poisson bracket between two Gauss' law constraints G_a,

$$\{\vec{G}[\vec{\theta}], \vec{G}[\vec{\lambda}]\} = \int_\Sigma d^3x \Big[\frac{\delta \vec{G}[\vec{\theta}]}{\delta \Psi_{bf}(x)} \frac{\delta \vec{G}[\vec{\lambda}]}{\delta X^{bf}(x)} - \frac{\delta \vec{G}[\vec{\lambda}]}{\delta \Psi_{bf}(x)} \frac{\delta \vec{G}[\vec{\theta}]}{\delta X^{bf}(x)} \Big]$$
$$= \int_\Sigma d^3x \Big[(\mathbf{w}_f\{\theta^b\})(\overline{W}^{ae}_{bf}(\vec{\lambda})\Psi_{ae}) - (\mathbf{w}_f\{\theta^b\})(\overline{W}^{ae}_{bf}(\vec{\theta})\Psi_{ae}) \Big]$$
$$\equiv W^{ae}(\vec{\theta}, \vec{\lambda}) \Psi_{ae} \qquad (64)$$

where $W^{ae}(\vec{\theta}, \vec{\lambda}) \equiv (\mathbf{w}_f\{\theta^g\}) W^{ae}_{bf}(\vec{\lambda})$ will be suitably defined through the following steps. We will interpret the Poisson bracket of two Gauss' law

constraints as a Gauss' law constraint in the following sense. Expanding the first term of (64) for illustrative purposes,

$$\int_\Sigma d^3x \frac{\delta\vec{G}[\vec{\theta}]}{\delta\Psi_{bf}(x)} \frac{\delta\vec{G}[\vec{\lambda}]}{\delta X^{bf}(x)} - \vec{\theta} \leftrightarrow \vec{\lambda}$$

$$= -\int_\Sigma d^3x (B^i_f D_i\theta^b)\Big((B^{-1})^f_l \overline{D}^{il}_{eb}(\lambda^a D_i\Psi_{ae}) + \lambda^a(f_{abh}\delta_{gf} + f_{fbg}\delta_{ah})\Psi_{hg}\Big)$$

$$= \int_\Sigma d^3x (D^{il}_{eb}\{D_l\theta^b\})\lambda^a D_i\Psi_{ae} - \int_\Sigma (B^i_f D_i\theta^b)\lambda^a(f_{abh}\Psi_{hf} + f_{fbg}\Psi_{ag}) \quad (65)$$

where we have integrated the first term by parts after using $B^i_f(B^{-1})^f_l = \delta^i_l$. Applying the definition of curvature as the commutator of covariant derivatives $D^{il}_{eb}D_l\theta^b = f_{ebc}B^i_b\theta^c$ on the first term of (65), we have

$$\int_\Sigma d^3x \Big[(f_{ebc}B^i_b\theta^c)\lambda^a D_i\Psi_{ae} - \lambda^a \mathbf{w}_f\{\theta^b\}(f_{abh}\Psi_{hf} + f_{fbg}\Psi_{ag}). \quad (66)$$

Upon subtraction of the contribution when λ^a and θ^a are reversed, we obtain

$$f_{ebc}(\theta^c\lambda^a - \lambda^c\theta^a)\mathbf{w}_b\{\Psi_{ae}\} + \big(\theta^a\mathbf{w}_f\{\lambda^b\} - \lambda^a\mathbf{w}_f\{\theta^b\}\big)(f_{abh}\Psi_{hf} + f_{fbg}\Psi_{ag}). (67)$$

To view (64) as a smeared Gauss' law constraint, we must separate the contribution to due to vector fields $\mathbf{v}_e$ from the contribution due to the tensor strucure of C_{be}. The first term of (67) expands to

$$\begin{aligned} f_{ebc}(\theta^c\lambda^a - \lambda^c\theta^a)\mathbf{w}_b\{\Psi_{ae}\} &= f_{ebc}(\theta^c\lambda^a - \lambda^c\theta^a)\mathbf{v}_b\{\Psi_{ae}\} \\ &+ f_{ebc}(\theta^c\lambda^a - \lambda^c\theta^a)C_{de}\big(f_{adh}\Psi_{he} + f_{edg}\Psi_{ag}\big), \end{aligned} \quad (68)$$

where we have used the definition of the covariant derivative of Ψ_{ae} in the tensor representation. Relabelling $e \to f$ and $b \to d$ on the second term of the right hand side of (68) and combining the result of (68) with the second term of (67) we obtain

$$f_{ebc}(\theta^c\lambda^a - \lambda^c\theta^a)\mathbf{v}_b\{\Psi_{ae}\}$$
$$+\Big(\lambda^a\big(f_{bce}\theta^c C_{de} - \mathbf{w}_e\{\theta^d\}\big) - \theta^a\big(f_{bce}\lambda^c C_{de} - \mathbf{w}_e\{\lambda^d\}\big)\Big)\big(f_{adh}\Psi_{he} + f_{edg}\Psi_{ag}\big). (69)$$

The standard form of the contracted Gauss' law generator is given by

$$\begin{aligned} &\theta^a\mathbf{v}_e\{\Psi_{ae}\} + \theta^a C^{fg}_a\Psi_{fg} \\ &= \theta^a\mathbf{v}_e\{\Psi_{ae}\} + \theta^a C_{be}\big(f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag}\big), \end{aligned} \quad (70)$$

where $C_{be} = A^b_i B^i_e$ is defined as the 'magnetic helicity density matrix'. If (69) is interpreted as a Gauss' law contraint defined with respect to composite parameters vector field and a transformed magnetic helicity, then the Gauss' law part of the algebra in the instanton representation closes into a subalgebra $\boldsymbol{A}_{gauge}(\Omega_{Inst})$.[16]

Moving on to the Poisson bracket between two spatial diffeomorphism constraints H_i, we have

$$\{\vec{H}[\vec{N}],\vec{H}[\vec{M}]\} = \int_\Sigma d^3x\Big[\frac{\delta\vec{H}[\vec{M}]}{\delta\Psi_{bf}(x)}\frac{\delta\vec{H}[\vec{N}]}{\delta X^{bf}(x)} - \frac{\delta\vec{H}[\vec{N}]}{\delta\Psi_{bf}(x)}\frac{\delta\vec{H}[\vec{M}]}{\delta X^{bf}(x)}\Big]$$
$$= 4\int_\Sigma d^3x\Big[(\vec{N}\times\vec{B})_{kb}B^k_f(B^{-1})^f_l\overline{D}^{jl}_{ab}((\vec{M}\times\vec{B})_{je}\Psi_{[ae]}) - \vec{M}\leftrightarrow\vec{N}\Big]. \tag{71}$$

Using $B^k_f(B^{-1})^f_l = \delta^k_l$ and integrating by parts, (71) yields

$$\{\vec{H}[\vec{N}],\vec{H}[\vec{M}]\} = 4\int_\Sigma d^3x\Big[(\vec{M}\times\vec{B})_{ke}D^{jk}_{ab}(\vec{N}\times B)_{jb} - N\leftrightarrow M\Big]\Psi_{[ae]}$$
$$\equiv V^{ae}(\vec{M},\vec{N})\Psi_{[ae]}. \tag{72}$$

The Poisson bracket of two diffeomorphism constraints is a diffeomorphism constraint, since it is also linear in the antisymmetric part of Ψ_{ae}. The composite parameter for the resulting diffeomorphism has the intuitively appealing form of a kind of covariant $SU(2)_-$ Lie derivative of $\vec{N}$ along $\vec{M}$, which involves their components orthogonal to the spatial 3-vectors $\vec{B}_a$. The result is that the spatial diffeomorphisms form their own subalgebra $\boldsymbol{A}_{diff}(\Omega_{Inst})$.

Next, we proceed to the Poisson bracket between two Hamiltonian constraints

$$\{H[N],H[M]\} = \int_\Sigma d^3x\Big[\frac{\delta H[N]}{\delta\Psi_{bf}(x)}\frac{\delta H[M]}{\delta X^{bf}(x)} - \frac{\delta H[M]}{\delta\Psi_{bf}(x)}\frac{\delta H[N]}{\delta X^{bf}(x)}\Big]$$
$$= \int_\Sigma d^3x\Big[((\det B)^{1/2}N\boldsymbol{M}^{bf})\Big(\frac{1}{2}(B^{-1})^f_l\overline{D}^{kl}_{ab}\big((B^{-1})^a_kH(M)\big)\Big)$$
$$-(\det B)^{1/2}M\boldsymbol{M}^{bf}\Big(\frac{1}{2}(B^{-1})^f_l\overline{D}^{kl}_{ab}\big((B^{-1})^a_kH(N)\big)\Big). \tag{73}$$

We are beginning to see the appearance of the momentum-dependent structure functions $\boldsymbol{M}^{bf}$, given by

[16]This is argued from the perspective that the aforementioned structures which define the transformations generated by G_a can be read off directly from (69), even if the composition law of its constituents in relation to (70) may be intricate.

$$\boldsymbol{M}^{bf}=(\text{det}B)^{-1/2}\Big(\big(\frac{1}{2}(\Psi^{-1})^{bf}H-(\text{det}B)^{1/2}(\Psi^{-1}\Psi^{-1})^{bf}\big)\Big) \tag{74}$$

as in (47). Integrating by parts and discarding boundary terms, we obtain

$$\frac{1}{2}\int_{\Sigma}d^3x(\text{det}B)^{1/2}(B^{-1})^a_k(B^{-1})^f_l\boldsymbol{M}^{bf}\epsilon^{klm}\big(M(D_m)_{ab}N-N(D_m)_{ab}M\big)H \tag{75}$$

where we have used the fact that M and N are scalars under the antisymmetry operation.[17] Continuing along,

$$\frac{1}{2}\int_{\Sigma}d^3x(\text{det}B)^{1/2}(B^{-1})^a_k(B^{-1})^f_l\boldsymbol{M}^{bf}\epsilon^{klm}\delta_{ab}\big(M\partial_mN-N'\partial_mM\big)H$$
$$=\frac{1}{2}\int_{\Sigma}d^3x(\text{det}B)^{1/2}(\text{det}B)^{-1}\epsilon^{afd}B^m_d\boldsymbol{M}^{af}\big(M\partial_mN'-N\partial_mM\big)H$$
$$=\frac{1}{2}\int_{\Sigma}d^3x(\text{det}B)^{-1/2}\epsilon^{afd}\boldsymbol{M}^{af}\big(M\mathbf{v}_d\{N\}-N\mathbf{v}_d\{M\}\big)H, \tag{76}$$

where we have defined vector fields $\mathbf{v}_a=B^i_a\partial_i$. The Poisson bracket of two Hamiltonian constraints is another Hamiltonian constraint with momentum dependent structure functions. However, note that ϵ^{afd} attempts to select the antisymmetric part of $\boldsymbol{M}^{af}$. If $\boldsymbol{M}^{af}$ were symmetric in a and f, then two Hamiltonian constraints in the instanton representation would strongly commute. If we define a $SU(2)_-$-valued internal three-vector $\mathbf{m}^d$ by

$$\mathbf{m}^d=\epsilon_{afd}\boldsymbol{M}^{af}(\text{det}B)^{-1/2}, \tag{77}$$

and the three-vector $V_d(M,N)=M\mathbf{v}_d\{N\}-N'\mathbf{v}_d\{M\}$, then we have

$$\{H[N],H[M]\}=\mathbf{m}^dV_d(M,N)H=(\vec{\mathbf{m}}\cdot\vec{V}(M,N))H. \tag{78}$$

Hence, the degree of noncommutativity of two normal deformations is related to the projection of the vector $\vec{\mathbf{m}}$ into the vector $\vec{V}$. The end result is that the Hamiltonian constraint forms its own subalgebra $\boldsymbol{A}_H(\Omega_{Inst})$ in the instanton representation.[18]

[17] In other words, the covariant derivatives can be replaced by noncovariant derivatives, since they act on scalars.

[18] This is in contrast to the Ashtekar and the metric variables, where the bracket between two Hamiltonian constraints is a diffeomorphism constraint. Hence the instanton representation can provide an alternative approach to the resolution of the representation theory, which for loop quantum gravity is one of the issues that the Master Constraint programme was designed to address [8].

4.3 Computation of the mixed Poisson brackets

Now that we have computed the Poisson brackets involving constraints of the same type, we now move on to Poisson, brakcets between constraints of different type. Starting with the Poisson bracket between a Gauss' law constraint G_a and a diffeomorphism constraint H_i we have

$$\{\vec{G}[\vec{\theta}],\vec{H}[\vec{N}]\} = \int_{\Sigma} d^3x \Big[\frac{\delta\vec{G}[\vec{\theta}]}{\delta\Psi_{bf}(x)}\frac{\delta\vec{H}[\vec{N}]}{\delta X^{bf}(x)} - \frac{\delta\vec{H}[\vec{N}]}{\delta\Psi_{bf}(x)}\frac{\delta\vec{G}[\vec{\theta}]}{\delta X^{bf}(x)}\Big]$$

$$= \int_{\Sigma} d^3x \Big[(B^i_f D_i\theta^b)\big(-4(B^{-1})^f_l \overline{D}^{jl}_{ab}((\vec{N}\times\vec{B})_{je}\Psi_{[ae]})\big) - (\vec{N}\times\vec{B})_{kb}B^k_f W^{ae}_{bf}(\vec{\theta})\Psi_{ae}\Big]$$

$$= \int_{\Sigma} d^3x \Big[-4D^{jl}_{ab}(D_l\theta^b)((\vec{N}\times\vec{B})_{je}\Psi_{[ae]}) - (\vec{N}\times\vec{B})_{kb}B^k_f W^{ae}_{bf}(\vec{\theta})\Psi_{ae}\Big], \quad (79)$$

where we have used $B^i_f(B^{-1})^f_l = \delta^i_l$ in conjunction with an integration by parts on the first term. Note that the part in brackets involving $\Psi_{[ae]}$ is no longer acted on by spatial gradients. Upon application of the definition of curvature as the commutator of covariant derivatives, the coefficient of this factor is given by

$$D^{jl}_{ab}D_l\theta^b = \epsilon_{abc}B^j_b\theta^c = -(\vec{\theta}\times\vec{B})^j_a \quad (80)$$

Hence the first part of the right hand side of (79) is given by

$$4\int_{\Sigma} d^3x(\vec{\theta}\times\vec{B})^j_a((\vec{N}\times\vec{B})_{je}\Psi_{[ae]}) \equiv V^{ae}(\vec{\theta},\vec{N})\Psi_{[ae]}, \quad (81)$$

which is clearly a diffeomorphism constraint with mixed parameters $\vec{\theta}\equiv\theta^a$ and $\vec{N}\equiv N^i$. The second term of (79) can be interpreted as a Gauss' law constraint, which can be seen as follows. First recall the definition of W^{ae}_{bf} from (44), relabelling the indices $h\leftrightarrow a$ and $g\leftrightarrow e$ on the first term. This is given by

$$W^{hg}_{bf}(\vec{\theta})\Psi_{hg} = (B^{-1})^f_l \overline{D}^{il}_{gb}(\theta^h D_i\Psi_{hg}) + \theta^a\big(f_{abh}\delta_{gf} + f_{fbg}\delta_{ah}\big)\Psi_{hg}. \quad (82)$$

We will now contract (82) with $(\vec{N}\times\vec{B})_{kb}B^k_f$ as in (79) followed by an integration by parts with discarding of boundary terms. The first contribution to (82) upon integration by parts reduces to the vector field structure acting as a divergence on Ψ_{hg}

$$(\vec{N}\times\vec{B})_{kb}B^k_f(B^{-1})^f_l\overline{D}^{ik}_{gb}(\theta^h D_i\Psi_{hg}) = (\vec{N}\times\vec{B})_{kb}\overline{D}^{ik}_{gb}(\theta^h D_i\Psi_{hg})$$
$$\longrightarrow D^{ik}_{gb}(\vec{N}\times\vec{B})_{kb}\theta^h D_i\Psi_{hg} \equiv \mathbf{v}^{hg}(\vec{N},\vec{\theta})\Psi_{hg}. \quad (83)$$

Note that this vector field $\mathbf{v}^{hg}(\vec{N},\vec{\theta})$ is now determined by mixed composite parameters $\vec{\theta}\equiv\theta^a$ and $\vec{N}\equiv N^i$. The second contribution to (82) involves the tensor representation of structure acting on Ψ_{hg}, namely

$$\int_\Sigma d^3x\epsilon_{ijk}N^iB^j_bB^k_f\theta^a\big(f_{abh}\delta_{gf}+f_{fbg}\delta_{ah}\big)\Psi_{hg}. \tag{84}$$

We will leave (84) in its present form, simply noting that it invovles the same tensor structure of (58), evaluated on mixed parameters $\vec{N}$ and $\vec{\theta}$. Therefore the combination of (84) with the result of (83) yields result of

$$\int_\Sigma d^3x\Big(\mathbf{v}^{hg}(\vec{N},\vec{\theta})+(\vec{N}\times\vec{B})_{kb}\theta^aB^k_f\big(f_{abh}\delta_{gf}+f_{fbg}\delta_{ah}\big)\Big)\Psi_{hg}$$
$$\equiv\int_\Sigma d^3xW^{hg}(\vec{N},\vec{\theta})\Psi_{hg} \tag{85}$$

as per the notation of (59). Combining (85) with (81), the end result is

$$\{\vec{G}[\vec{\theta}],\vec{H}[\vec{N}]\}=V^{ae}(\vec{N},\vec{\theta})\Psi_{[ae]}-W^{ae}(\vec{N},\vec{\theta})\Psi_{ae}. \tag{86}$$

Equation (86) states that the Possion bracket between a Gauss' law constraint G_a and a diffeomorphism constraint H_i yields a linear combination of the two constraints with mixed parameters. Hence from equation (86), in conjunction with the results of (64) and (71), one sees that the kinematic constraints in the instanton representation form their own algebra $\boldsymbol{A}_{Kin}(\Omega_{Inst})=\boldsymbol{A}_{gauge}\times\boldsymbol{A}_{diff}$, in a kind of semidirect product structure. This is similar to the case in the Ashtekar variables.

We now compute the Poisson bracket between a diffeomorphism constraint H_i and a Hamiltonian constraint H

$$\{H[N],\vec{H}[\vec{N}]\}=\int_\Sigma d^3x\Big[\frac{\delta H[N]}{\delta\Psi_{bf}(x)}\frac{\delta\vec{H}[\vec{N}]}{\delta X^{bf}(x)}-\frac{\delta\vec{H}[\vec{N}]}{\delta\Psi_{bf}(x)}\frac{\delta H[N]}{\delta X^{bf}(x)}\Big]$$
$$\int_\Sigma d^3x\Big[N(\det B)^{1/2}\boldsymbol{M}^{bf}\Big(-4(B^{-1})^f_l\overline{D}^{jl}_{ab}((\vec{N}\times\vec{B})_{je}\Psi_{[ae]})\Big)$$
$$-(\vec{N}\times\vec{B})_{kb}B^k_f\Big(\frac{1}{2}(B^{-1})^f_l\overline{D}^{ml}_{ab}((B^{-1})^a_mHN)\Big). \tag{87}$$

Integration of the first term of (87) by parts transfers the derivative away from $\Psi_{[ae]}$, yielding

$$-4\int_\Sigma d^3xD^{jl}_{ab}\big(N(\det B)^{1/2}\boldsymbol{M}^{bf}(B^{-1})^f_l\big)(\vec{N}\times\vec{B})_{je}\Psi_{[ae]}\equiv\boldsymbol{V}^{ae}(\vec{N},N)\Psi_{[ae]}. \tag{88}$$

The result of (88) is categorized as a diffeomorphism constraint since a factor linear in $\Psi_{[ae]}$ has been isolated. The boldface $\boldsymbol{V}^{ae}(N\vec{N})$ signifies that there are momentum-dependent structure functions $\boldsymbol{M}^{bf}$, smeared with composite parameters N and $\vec{N}=N^i$.[19]

Integration of the second term of (87) by parts yields

$$-\frac{1}{2}\int_\Sigma d^3xN(B^{-1})^a_mD^{mk}_{ab}((\vec{N}\times\vec{B})_{kb})H\equiv S_{Diff}[N,\vec{N}]H\equiv H[N,\vec{N}],\quad(89)$$

which is clearly a Hamitlonian constraint. The quantity S_{Diff} smearing the Hamiltonian constraint arose from fully contracting all spatial and $SU(2)_-$ indices as in (89), and should be a scalar with respect to the kinematic gauge subgroup. But is seems appropriate for notational purposes to absorb this whole quantity into an overall lapse function labelled by N and $\vec{N}$, hence the definition in (89). The end result is that the Hamiltonian and diffeomorphism constraints $H_\mu=(H,H_i)$ form their own subalgebra $\boldsymbol{A}_{Diff}(\Omega_{Inst})$. Combining the results of (89) with (88) we have

$$\{H[N],\vec{H}[\vec{N}]\}=\boldsymbol{V}^{ae}(N,\vec{N})\Psi_{[ae]}+H[N,\vec{N}],\quad(90)$$

namely that the Poisson bracket between a Hamiltonian constraint and a diffeomorphism constraint is a linear combination of the two constraints, with momentum-dependent structure functions. Equation (90), in conjunction with the results of (72) and (78), show that in the instanton representation the Hamiltonian and the diffeomorphism constraints form their own algebra. This is the case in the Ashtekar variables as well as in the metric variables, although the structure of the algebra is slightly different.

The Poisson bracket between a Hamiltonian constraint H and a Gauss' law constraint G_a is given by

$$\{H[N'],\vec{G}[\vec{\theta}]\}=\int_\Sigma d^3x\Big(\frac{\delta H[N]}{\delta\Psi_{bf}(x)}\frac{\delta\vec{G}[\vec{\theta}]}{\delta X^{bf}(x)}-\frac{\delta\vec{G}[\vec{\theta}]}{\delta\Psi_{bf}(x)}\frac{\delta H[N]}{\delta X^{bf}(x)}\Big)$$

$$\int_\Sigma d^3x\Big((\det B)^{1/2}N\boldsymbol{M}^{bf}W^{hg}_{bf}(\vec{\theta})\Psi_{hg}-(B^i_fD_i\theta^b)\Big(\frac{1}{2}(B^{-1})^f_l\overline{D}^{kl}_{ab}((B^{-1})^a_kHN)\Big).\quad(91)$$

The second term of (91) upon integration by parts induces the replacement $\overline{D}^{kl}_{ab}\rightarrow D^{kl}_{ab}$, which simplifies to

[19]Note that $\boldsymbol{M}^{bf}$ contains nonlinear dependence on $\Psi_{[ae]}$, and hence the full expression is in this sense not linear in $\Psi_{[ae]}$. But this nonlinear dependence is attributed to the structure functions whose coefficient, the part linear in $\Psi_{[ae]}$, is associated with the diffeomorphism constraint.

$$-\frac{1}{2}\int_\Sigma d^3x\epsilon^{klm}(D_m)_{ab}(D_l\theta^b)[(B^{-1})^a_k HN] = \frac{1}{2}\int_\Sigma d^3x(f_{adc}B^k_d\theta^c)(B^{-1})^a_k NH = 0 \quad (92)$$

where we have used $B^i_f(B^{-1})^f_l = \delta^i_l$ as well as the definition of curvature as the commutator of covariant derivatives, as well as antisymmetry of the structure constants f_{abc}. The first term on the right hand side of (91) can be computed using (82), which we repeat here

$$\overline{W}^{hg}_{bf}(\vec{\theta})\Psi_{hg} = (B^{-1})^f_l\overline{D}^{il}_{gb}(\theta^h D_i\Psi_{hg}) + \theta^a\big(f_{abh}\delta_{gf} + f_{fbg}\delta_{ah}\big)\Psi_{hg}. \quad (93)$$

This will yield two contributions, one which upon integration by parts will reduce to a vector field which contracts one index of Ψ_{hg}. The second part reduces to a transformation of Ψ_{hg} as a second-rank tensor. Integrating by parts, we obtain the first contribution

$$\int_\Sigma d^3x(\mathrm{det}B)^{1/2}N\boldsymbol{M}^{bf}(B^{-1})^f_l\overline{D}^{il}_{gb}(\theta^h D_i\Psi_{hg})$$
$$= D^{il}_{gb}\big((\mathrm{det}B)^{1/2}N\boldsymbol{M}^{bf}(B^{-1})^f_l\big)\theta^h D_i\Psi_{hg} \equiv \int_\Sigma d^3x\mathbf{V}^{hg}(N,\vec{\theta})\Psi_{hg}, \quad (94)$$

which involves the vector field $\boldsymbol{V}^{hg}(N,\vec{\theta})$ labelled by composite parameters N and $\vec{\theta}$.[20] The second contribution is given by

$$\int_\Sigma d^3x(\mathrm{det}B)^{1/2}N\boldsymbol{M}^{bf}\theta^a\big(f_{abh}\delta_{gf} + f_{fbg}\delta_{ah}\big)\Psi_{hg}. \quad (95)$$

The combination of (94) with (95) yields

$$\{H[N],\vec{G}[\vec{\theta}]\}$$
$$= \int_\Sigma\Big(\mathbf{V}^{hg}(N,\vec{\theta}) + (\mathrm{det}B)^{1/2}N\theta^a\boldsymbol{M}^{bf}\big(f_{abh}\delta_{gf} + f_{fbg}\delta_{ah}\big)\Big)\Psi_{hg}$$
$$\equiv \boldsymbol{W}^{ae}(N,\vec{\theta})\Psi_{ae}, \quad (96)$$

which can be classified as a smeared Gauss' law constraint with mixed parameters according to the notation of (59). The final result of (91) is

$$\{H[N],\vec{G}[\vec{\theta}]\} = \boldsymbol{W}^{ae}(N,\vec{\theta})\Psi_{ae} \quad (97)$$

[20] It is hopefully clear from the context that this is different from $\boldsymbol{V}^{ae}$ in (90), since $\mathbf{V}^{ae}$ is designed to reflect the fact that it is a vector field containing spatial gradients that act on Ψ_{ae}. It is also different from $\mathbf{v}^{ae}$ in (83) in that now there are momentum-dependent structure functions involved.

which states that a Gauss' law constraint transforms covariantly under Hamiltonian evolution. Another way to state this is that the Hamiltonian constraint on the full phase space Ω_{Inst} is not gauge invariant in the instanton representation, and also that momentum dependent structure functions appear in the Poisson bracket. This is in contrast to the case in Ashtekar variables, where the two constraints commute.

5 Summary and Recapitulation

The main results of this paper are as follows. We have transformed the action for general relativity from the Ashtekar variables $(\widetilde{\sigma}^i_a, A^a_i)_{HH}$ into the instanton representation, of phase space structure $(\Psi_{ae}, X^{ae})_{NH} \sim (\Psi_{ae}, A^a_i)_{HN}$. This is a noncanonical transformation, since the symplectic two form $\boldsymbol{\Omega}_{Inst}$ for the latter representation is not the total exterior derivative of its corresponding canonical one form $\boldsymbol{\theta}_{Inst}$.[21] The noncanonical relationship of phase space Ω_{Inst} to Ω_{Ash} does not preclude the ability to formulate Poisson brackets in the former. We have used this feature to evaluate the algebra of constraints in the instanton representation, and we have demonstrated the closure of this algebra. The algebra of constraints is given by

$$\begin{aligned}
\{\Psi_{[ae]}[V^{ae}(\vec{N})], \Psi_{[bf]}[V^{bf}(\vec{M})]\} &= \Psi_{[ae]}[V^{ae}(\vec{N}, \vec{M})];\\
\{\Psi_{[ae]}[V^{ae}(\vec{N})], \Psi_{bf}[W^{bf}(\vec{\theta})]\} &= \Psi_{[hg]}[V^{hg}(\vec{\theta}, \vec{N})] + \Psi_{hg}[W^{hg}(\vec{\theta}, \vec{N})];\\
\{\Psi_{ae}[W^{ae}(\vec{\theta})], \Psi_{bf}[W^{bf}(\vec{\lambda})]\} &= \Psi_{ae}[W^{ae}(\vec{\theta}, \vec{\lambda})];\\
\{H[N], \Psi_{[ae]}[V^{ae}(\vec{N})]\} &= \boldsymbol{V}^{ae}(\vec{N}, N)\Psi_{[ae]} + H[N, \vec{N}];\\
\{H[N], \Psi_{ae}[W^{ae}(\vec{\theta})]\} &= \boldsymbol{W}^{ae}(N, \vec{\theta})\Psi_{ae};\\
\{H[M], H[N]\} &= \vec{\mathbf{m}} \cdot \vec{V}(M, N)H.
\end{aligned} \tag{98}$$

Let us rewrite (98) in the following standardized notation

$$\begin{aligned}
\{\vec{H}[\vec{N}], \vec{H}[\vec{M}]\} &\sim \vec{H}[\vec{N}, \vec{M}];\\
\{\vec{H}[\vec{N}], \vec{G}[\vec{\theta}]\} &\sim \vec{H}[\vec{N}, \vec{\theta}] + \vec{G}[\vec{N}, \vec{\theta}];\\
\{\vec{G}[\vec{\theta}], \vec{G}[\vec{\lambda}]\} &\sim \vec{G}[\vec{\theta}, \vec{\lambda}];\\
\{H[N], \vec{H}[\vec{N}]\} &\sim \vec{H}[\vec{N}, N] + H[\vec{N}, N];\\
\{H[N], \vec{G}[\vec{\theta}]\} &\sim \vec{G}[N, \vec{\theta}];\\
\{H[M], H[N]\} &\sim H[M, N],
\end{aligned} \tag{99}$$

and compare and contrast the constraint algebra in instanton representation with that in the Ashtekar variables, which we quote below for completeness

[21] This is when one considers the full phase space. It is shown in subsequent papers in this series that for certain restricted configurations, one can obtain $\boldsymbol{\Omega}_{Inst} = \delta\boldsymbol{\theta}_{Inst}$ by using densitized variables as the fundamental variables.

$$
\begin{aligned}
\{\vec{H}[\vec{N}],\vec{H}[\vec{M}]\} &= H_k\big[N^i\partial^k M_i - M^i\partial^k N_i\big]\\
\{\vec{H}[N],G_a[\theta^a]\} &= G_a[N^i\partial_i\theta^a]\\
\{G_a[\theta^a],G_b[\lambda^b]\} &= G_a\big[f^a_{bc}\theta^b\lambda^c\big]\\
\{H(\underline{N}),\vec{H}[\vec{N}]\} &= H[N^i\partial_i\underline{N}]\\
\{H(\underline{N}),G_a(\theta^a)\} &= 0\\
\big[H(\underline{N}),H(\underline{M})\big] &= H_i[\big(\underline{N}\partial_j\underline{M}-\underline{M}\partial_j\underline{N}\big)H^{ij}].
\end{aligned}
\tag{100}
$$

The algebra of kinematic constraints $\boldsymbol{A}_{Kin}(\Omega_{Ash})$ in the Ashtekar variables, a semi-direct product of $SU(2)$ with spatial diffeomorphisms, is a Lie algebra. The inclusion of the Hamiltoninan constraint enlarges the kinematic algebra into an open algebra due to the structure functions [2],[3].

There are some similarities and some differences between the algebraic structure of (99) and (100). (i) There are three subalgebras within the algebraic structure (99). Spatial diffeomorphisms H_i and Gauss' law G_a each form independent subalgebras as for (100). However, the Hamiltonian constraint H now forms its own independent subalgebra. Also note the structure $[A,A]\sim A$, and $[A,B]\sim A+B$, where A and B are transformations of different type. (ii) In (100) the structure functions occur only in the Poisson bracket between two Hamiltonian constraints, whereas in (99) momentum-dependent structure functions $\boldsymbol{M}^{bf}$ appear in any Poisson bracket with the Hamiltonian constraint H. The parameters of the transformations generated by (99) contain field dependence, but there is a difference between configuration space dependence and momentum space dependence. The former can in a sense be interpreted as part of the definition of the gauge parameters of the transformation, but the latter is regarded as a new fundamental structure induced by the Hamiltonian constraint H. Recall from [9] that the structure functions of the hypersurface deformation algebra are related to the induced 3-metric h_{ij} on the spatial hypersurface Σ. The analogue of this in the instanton representation is the internal metric η^{bf} of (74), from which h_{ij} is a derived object through the relation (49).
(iii) Lastly, in (100) all constraints transform covariantly under diffeomorphisms whereas in (99) the transformations appear on a more equal footing. On this basis, we surmize that the instanton representation should provide aspects of the dynamics of gravity which are different from those in the Ashtekar variables.

(iv) Lastly, the Hamiltonian constraint part of (99) closes, unlike in (100), and therefore by itself forms a first class system. This implies in the instanton representation that it is possible to implement the kinematic constraints, obtaining a reduced phase space in the full theory consistently governed by the dynamics of the Hamiltonian constraint.[22] From this per-

[22]This is shown both for the classical and for the quantum theory in the instanton

spective, one may eliminate the kinematic constraints using Dirac brackets, leaving behind just the physical degrees of freedom.

The main result of this paper is that the classical algebra of constraints on the instanton representation phase space $\Omega_{Inst} = (\Psi_{ae}, A^a_i)$ closes in the sense that we have explained in this paper. This means that the theory is Direac consistent and as well amenable to a quantization on its reduced phase space. This will form the basis for progressing to the quantum theory, which is treated within the instanton representation series (See e.g. Paper II for the listing of this series).

representation series of papers.

6 Appendix A. Expansion of the instanton terms

Let us perform a 3+1 decomposition of the action, which contains terms of the form $F^a_{0i}F^e_{jk}\epsilon^{ijk}$. Expanding the time component, we have

$$F^a_{0i}F^e_{jk}\epsilon^{ijk} = B^i_e\big(\partial_0 A^a_i - \partial_i A^a_0 + f^{abc}A^b_0 A^c_i\big) = B^i_e\dot{A}^a_i - B^i_e\partial_i A^a_0 + f^{abc}A^b_0 C_{ce}, \quad (101)$$

where we have defined $C_{be} = A^b_i B^i_e$. Now contract (101) with $\Psi_{ae} = \Psi_{(ae)}$

$$\frac{1}{2}\Psi_{ae}F^a_{\mu\nu}F^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = \Psi_{ae}B^i_e\dot{A}^a_i - \Psi_{ae}B^i_e\partial_i A^a_0 + f^{abc}A^b_0 C_{ce}\Psi_{ae}$$
$$= \Psi_{ae}B^i_e\dot{A}^a_i - \partial_i(\Psi_{ae}B^i_e A^a_0) + A^a_0\partial_i(\Psi_{ae}B^i_e) + f^{abc}A^b_0 C_{ce}\Psi_{ae}. \quad (102)$$

Applying the Leibniz rule, the right hand side of (102) reduces to[23]

$$\Psi_{ae}B^i_e\dot{A}^a_i + A^a_0\big(B^i_e\partial_i\Psi_{ae} + \Psi_{ae}\partial_i B^i_e\big) + f^{abc}A^b_0 C_{ce}\Psi_{ae}. \quad (103)$$

Next, we make use of the Bianchi identity to write the divergence of B^i_e in terms of a covariant divergence and simplify further.[24]

$$D_i B^i_e = \partial_i B^i_e + f_{egh}A^g_i B^i_h = 0 \rightarrow \partial_i B^i_e = f_{ehg}C_{gh}. \quad (104)$$

Substituting (104) into (103) we obtain

$$\Psi_{ae}B^i_e\dot{A}^a_i + A^a_0 B^i_e\partial_i\Psi_{ae} + A^a_0\Psi_{ae}f_{ehg}C_{gh} + f_{abc}A^b_0 C_{ce}\Psi_{ae}. \quad (105)$$

Relabelling indices $a \leftrightarrow b$ on the last term and $b \leftrightarrow f$ and $e \leftrightarrow g$ on the middle two terms of (105), we obtain

$$\Psi_{ae}B^i_e\dot{A}^a_i + A^a_0\Big(B^i_e\partial_i\Psi_{ae} + \big(f_{ghe}\delta_{af} + f_{fae}\delta_{gh}\big)C_{eh}\Psi_{fg}\Big). \quad (106)$$

Now make the following definition

$$\mathbf{w}_e\{\Psi_{ae}\} = B^i_e\partial_i\Psi_{ae} + \big(f_{ghe}\delta_{af} + f_{fae}\delta_{gh}\big)C_{eh}\Psi_{fg} \equiv \mathbf{w}^{fg}_a\{\Psi_{fg}\}. \quad (107)$$

[23]Omitting the total derivative, since it will integrate to a boundary term in the action

[24]The gauge connection A^a_i in (104) is precisely the connection from which the magnetic field $B^i_a = \epsilon^{ijk}\partial_j A^a_k + \frac{1}{2}\epsilon^{ijk}f^{abc}A^b_j A^c_k$ is derived. Hence the Bianchi identity is a highly nonlinear constraint upon B^i_a. Moreover, the quantity F^a_{0i} expressed in terms of B^i_a is also highly nonlinear, and contains three arbitrary degrees of freedom encoded in A^a_0.

We will see that $\mathbf{w}_a^{fg}\{\Psi_{fg}\} = B^i_e D_i \Psi_{ae}$ is the covariant divergence of Ψ_{ae} in the tensor representation of the gauge group. Integrating over spacetime M, we obtain

$$\int_M d^4x \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = \int_0^T \int_\Sigma d^3x \big(\Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 \mathbf{w}_e\{\Psi_{ae}\} \big). \quad (108)$$

The resulting action is a totally constrained system consisting of the constraint $\mathbf{w}_e\{\Psi_{ae}\} \sim 0$.

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